

AN UPPER BOUND ON THE ABBES-SAITO FILTRATION FOR FINITE FLAT GROUP SCHEMES AND APPLICATIONS

YICHAO TIAN

ABSTRACT. Let \mathcal{O}_K be a complete discrete valuation ring of residue characteristic $p > 0$, and G be a finite flat group scheme over \mathcal{O}_K of order a power of p . We prove in this paper that the Abbes-Saito filtration of G is bounded by a linear function of the degree of G . Assume \mathcal{O}_K has generic characteristic 0 and the residue field of \mathcal{O}_K is perfect. Fargues constructed the higher level canonical subgroups for a “not too supersingular” Barsotti-Tate group \mathcal{G} over \mathcal{O}_K . As an application of our bound, we prove that the canonical subgroup of \mathcal{G} of level $n \geq 2$ constructed by Fargues appears in the Abbes-Saito filtration of the p^n -torsion subgroup of \mathcal{G} .

Let \mathcal{O}_K be a complete discrete valuation ring with residue field k of characteristic $p > 0$ and fraction field K . We denote by v_π the valuation on K normalized by $v_\pi(K^\times) = \mathbf{Z}$. Let G be a finite and flat group scheme over \mathcal{O}_K of order a power of p such that $G \otimes K$ is étale. We denote by $(G^a, a \in \mathbf{Q}_{\geq 0})$ the Abbes-Saito filtration of G . This is a decreasing and separated filtration of G by finite and flat closed subgroup schemes. We refer the readers to [AS02, AS03, AM04] for a full discussion, and to section 1 for a brief review of this filtration. Let ω_G be the module of invariant differentials of G . The generic étaleness of G implies that ω_G is a torsion \mathcal{O}_K -module of finite type. There exist thus nonzero elements $a_1, \dots, a_d \in \mathcal{O}_K$ such that

$$\omega_G \simeq \bigoplus_{i=1}^d \mathcal{O}_K / (a_i).$$

We put $\deg(G) = \sum_{i=1}^d v_\pi(a_i)$, and call it the degree of G . The aim of this note is to prove the following

Theorem 0.1. *Let G be a finite and flat group scheme over \mathcal{O}_K of order a power of p such that $G \otimes K$ is étale. Then we have $G^a = 0$ for $a > \frac{p}{p-1} \deg(G)$.*

Our bound is quite optimal when G is killed by p . Let $E_\delta = \text{Spec}(\mathcal{O}_K[X]/(X^p - \delta X))$ be the group scheme of Tate-Oort over \mathcal{O}_K . We have $\deg(E_\delta) = v_\pi(\delta)$, and an easy computation by Newton polygons gives [Fa09, Lemme 5]

$$E_\delta^a = \begin{cases} E_\delta & \text{if } 0 \leq a \leq \frac{p}{p-1} \deg(E_\delta) \\ 0 & \text{if } a > \frac{p}{p-1} \deg(E_\delta). \end{cases}$$

Date: May 16th, 2010.

However, our bound may be improved when G is not killed by p or G contains many identical copies of a closed subgroup. In [Hat06, Thm. 7], Hattori proves that if K has characteristic 0 and G is killed by p^n , then the Abbes-Saito filtration of G is bounded by that of the multiplicative group μ_{p^n} , *i.e.*, we have $G^a = 0$ if $a > en + \frac{e}{p-1}$ where e is the absolute ramification index of K . Compared with Hattori's result, our bound has the advantage that it works in both characteristic 0 and characteristic p , and that it is good if $\deg(G)$ is small.

The basic idea to prove 0.1 is to approximate general power series over \mathcal{O}_K by linear functions. First, we choose a “good” presentation of the algebra of G such that the defining equations of G involve only terms of total degree $m(p-1) + 1$ with $m \in \mathbf{Z}_{\geq 0}$ (Prop. 1.5). The existence of such a presentation is a consequence of the classical theory on p -typical curves of formal groups. With this good presentation, we can prove that the neutral connected component of the a -tubular neighborhood of G is isomorphic to a closed rigid ball for $a > \frac{p}{p-1} \deg(G)$ (Lemma 1.7), and the only zero of the defining equations of G in the neutral component is the unit section.

The motivation of our theorem comes from the theory of canonical subgroups. We assume that K has characteristic 0, and the residue field k is perfect of characteristic $p \geq 3$. Let G be a Barsotti-Tate group of dimension $d \geq 1$ over \mathcal{O}_K . If G comes from an abelian scheme over A , the canonical subgroup of level 1 of G was first constructed by Abbes and Mokrane in [AM04]. Then the author generalized their result to the Barsotti-Tate case [Ti06]. We actually proved that if a Barsotti-Tate group G over \mathcal{O}_K is “not too supersingular”, a condition expressed explicitly as a bound on the Hodge height of G (cf. 2.1), then a certain piece of the Abbes-Saito filtration of $G[p]$ lifts the kernel of Frobenius of the special fiber of G [Ti06, Thm. 1.4]. Later on, Fargues [Fa09] gave another construction of the canonical subgroup of level 1 by using Hodge-Tate maps, and his approach also allowed us to construct by induction the canonical subgroups of level $n \geq 2$, *i.e.*, the canonical lifts of the kernel of n -th iteration of the Frobenius. He proved that the canonical subgroup of higher level appears in the Harder-Narasimhan filtration of $G[p^n]$, which was introduced by him in [Fa07]. It is conjectured that the canonical subgroup of higher level also appears in the Abbes-Saito filtration of $G[p^n]$. In this paper, we prove this conjecture as a corollary of 0.1 (Thm. 2.5). We use essentially the result of Fargues on the degree of the quotient of $G[p^n]$ by its canonical subgroup of level n (see Thm. 2.4(i)).

0.2. Notation. In this paper, \mathcal{O}_K will denote a complete discrete valuation ring with residue field k of characteristic $p > 0$, and with fraction field K . Let π be a uniformizer of \mathcal{O}_K , and v_π be the valuation on K normalized by $v_\pi(\pi) = 1$. Let \overline{K} be an algebraic closure of K , K^{sep} be the separable closure of K contained in \overline{K} , and \mathcal{G}_K be the Galois group $\text{Gal}(K^{\text{sep}}/K)$. We denote still by v_π the unique extension of the valuation to \overline{K} .

1. PROOF OF THEOREM 0.1

We recall first the definition of the filtration of Abbes-Saito for finite flat group schemes according to [AM04, AS03].

1.1. For a semi-local ring R , we denote by \mathfrak{m}_R its Jacobson radical. An algebra R over \mathcal{O}_K is called *formally of finite type*, if R is semi-local, complete with respect to the \mathfrak{m}_R -adic topology, Noetherian and R/\mathfrak{m}_R is finite over k . We say an \mathcal{O}_K -algebra R formally of finite type is formally smooth, if each of the factors of R is formally smooth over \mathcal{O}_K .

Let $\mathbf{FEA}_{\mathcal{O}_K}$ be the category of finite, flat and generically étale \mathcal{O}_K -algebras, and $\mathbf{Set}_{\mathcal{G}_K}$ be the category of finite sets endowed with a continuous action of the Galois group \mathcal{G}_K . We have the fiber functor

$$\mathcal{F} : \mathbf{FEA}_{\mathcal{O}_K} \rightarrow \mathbf{Set}_{\mathcal{G}_K},$$

which associates with an object A of $\mathbf{FEA}_{\mathcal{O}_K}$ the set $\mathrm{Spec}(A)(\overline{K})$ equipped with the natural action of \mathcal{G}_K . We define a filtration on the functor \mathcal{F} as follows. For each object A in $\mathbf{FEA}_{\mathcal{O}_K}$, we choose a presentation

$$(1.1.1) \quad 0 \rightarrow I \rightarrow \mathcal{A} \rightarrow A \rightarrow 0,$$

where \mathcal{A} is an \mathcal{O}_K -algebra formally of finite type and formally smooth. For any $a = \frac{m}{n} \in \mathbf{Q}_{>0}$ with m prime to n , we define \mathcal{A}^a to be the π -adic completion of the subring $\mathcal{A}[I^n/\pi^m] \subset \mathcal{A} \otimes_{\mathcal{O}_K} K$ generated over \mathcal{A} by all the f/π^m with $f \in I^n$. The \mathcal{O}_K -algebra \mathcal{A}^a is topologically of finite type, and the tensor product $\mathcal{A}^a \otimes_{\mathcal{O}_K} K$ is an affinoid algebra over K [AS03, Lemma 1.4]. We put $X^a = \mathrm{Sp}(\mathcal{A}^a \otimes_{\mathcal{O}_K} K)$, which is a smooth affinoid variety over K [AS03, Lemma 1.7]. We call it the *a-th tubular neighborhood of $\mathrm{Spec}(A)$ with respect to the presentation (1.1.1)*. The \mathcal{G}_K -set of the geometric connected components of X^a , denoted by $\pi_0(X^a(A)_{\overline{K}})$, depends only on the \mathcal{O}_K -algebra A and the rational number a , but not on the choice of the presentation [AS03, Lemma 1.9.2]. For rational numbers $b > a > 0$, we have natural inclusions of affinoid varieties $\mathrm{Sp}(A \otimes_{\mathcal{O}_K} K) \hookrightarrow X^b \hookrightarrow X^a$, which induce natural morphisms $\mathrm{Spec}(A)(\overline{K}) \rightarrow \pi_0(X^b(A)_{\overline{K}}) \rightarrow \pi_0(X^a(A)_{\overline{K}})$. For a morphism $A \rightarrow B$ in $\mathbf{FEA}_{\mathcal{O}_K}$, we can choose properly presentations of A and B so that we have a functorial map $\pi_0(X^a(B)_{\overline{K}}) \rightarrow \pi_0(X^a(A)_{\overline{K}})$. Hence we get, for any $a \in \mathbf{Q}_{>0}$, a (contravariant) functor

$$\mathcal{F}^a : \mathbf{FEA}_{\mathcal{O}_K} \rightarrow \mathbf{Set}_{\mathcal{G}_K}$$

given by $A \mapsto \pi_0(X^a(A)_{\overline{K}})$. We have natural morphisms of functors $\phi_a : \mathcal{F} \rightarrow \mathcal{F}^a$, and $\phi_{a,b} : \mathcal{F}^b \rightarrow \mathcal{F}^a$ for rational numbers $b > a > 0$ with $\phi_a = \phi_{b,a} \circ \phi_b$. For any A in $\mathbf{FEA}_{\mathcal{O}_K}$, we have $\mathcal{F}(A) \xrightarrow{\sim} \varprojlim_{a \in \mathbf{Q}_{>0}} \mathcal{F}^a(A)$ [AS02, 6.4]; if A is a complete intersection over \mathcal{O}_K , the map $\mathcal{F}(A) \rightarrow \mathcal{F}^a(A)$ is surjective for any a [AS02, 6.2].

1.2. Let $G = \mathrm{Spec}(A)$ be a finite and flat group scheme over \mathcal{O}_K such that $G \otimes K$ is étale over K , and $a \in \mathbf{Q}_{>0}$. The group structure of G induces a group structure on $\mathcal{F}^a(A)$, and the natural map $G(\overline{K}) = \mathcal{F}(A) \rightarrow \mathcal{F}^a(A)$ is a homomorphism of groups. Hence the kernel $G^a(\overline{K})$ of $G(\overline{K}) \rightarrow \mathcal{F}^a(A)$ is a \mathcal{G}_K -invariant subgroup of $G(\overline{K})$, and it defines a closed subgroup scheme G_K^a of the generic fiber $G \otimes K$. The scheme theoretic closure of G_K^a in G , denoted by G^a , is a closed subgroup of G finite and flat over \mathcal{O}_K . Putting $G^0 = G$, we get a decreasing and separated filtration $(G^a, a \in \mathbf{Q}_{\geq 0})$ of G by finite and flat closed subgroup schemes. We call it *Abbes-Saito filtration* of G . For any real number $a \geq 0$, we put $G^{a+} = \cup_{b \in \mathbf{Q}_{>a}} G^b$.

Assume G is connected, *i.e.*, the ring A is local. Let

$$(1.2.1) \quad 0 \rightarrow I \rightarrow \mathcal{O}_K[[X_1, \dots, X_d]] \rightarrow A \rightarrow 0$$

be a presentation of A by the ring of formal power series such that the unit section of G corresponds to the point $(X_1, \dots, X_d) = (0, \dots, 0)$. Since A is a relative complete intersection over \mathcal{O}_K , I is generated by d elements f_1, \dots, f_d . For $a \in \mathbf{Q}_{>0}$, the \overline{K} -valued points of the a -th tubular neighborhood of G are given by

$$(1.2.2) \quad X^a(\overline{K}) = \{(x_1, \dots, x_d) \in \mathfrak{m}_{\overline{K}}^d \mid v_\pi(f_i(x_1, \dots, x_d)) \geq a \text{ for } 1 \leq i \leq d\},$$

where $\mathfrak{m}_{\overline{K}}$ is the maximal ideal of $\mathcal{O}_{\overline{K}}$. The subset $G(\overline{K}) \subset X^a(\overline{K})$ corresponds to the zeros of the f_i 's. Let X_0^a be the connected component of X^a containing 0. Then the subgroup $G^a(\overline{K})$ is the intersection of $X_0^a(\overline{K})$ with $G(\overline{K})$.

The basic properties of Abbes-Saito filtration that we need are summarized as follows.

Proposition 1.3 ([AM04] 2.3.2, 2.3.5). *Let G and H be finite and flat group schemes, generically étale over \mathcal{O}_K , $f : G \rightarrow H$ be a homomorphism of group schemes.*

- (i) G^{0+} is the connected component of G , and we have $(G^{0+})^a = G^a$ for any $a \in \mathbf{Q}_{>0}$.
- (ii) For $a \in \mathbf{Q}_{>0}$, f induces a canonical homomorphism $f^a : G^a \rightarrow H^a$. If f is flat and surjective, then $f^a(\overline{K}) : G^a(\overline{K}) \rightarrow H^a(\overline{K})$ is surjective.

Now we return to the proof of Theorem 0.1.

Lemma 1.4. *Let R be a \mathbf{Z}_p -algebra, \mathcal{X} be a formal group of dimension d over R . Then*

- (i) *the ring \mathbf{Z}_p acts naturally on \mathcal{X} , and its image in $\text{End}_R(\mathcal{X})$ lies in the center of $\text{End}_R(\mathcal{X})$;*
- (ii) *there exist parameters (X_1, \dots, X_d) of \mathcal{X} , such that we have $[\zeta](X_1, \dots, X_d) = (\zeta X_1, \dots, \zeta X_d)$ for any $(p-1)$ -th root of unity $\zeta \in \mathbf{Z}_p$.*

Proof. This is actually a classical result on formal groups. In the terminology of [Haz78], \mathcal{X} is necessarily isomorphic to a p -typical formal group over R [Haz78, 16.4.14]. This means that \mathcal{X} is deduced by base change from the universal p -typical formal group $\mathcal{X}^{\text{univ}}$ (denoted by $F_V(X, Y)$ in [Haz78, 15.2.8]) over $\mathbf{Z}_p[V] = \mathbf{Z}_p[V_i(j, k); i \in \mathbf{Z}_{\geq 0}, j, k = 1, \dots, d]$, where the $V_i(j, k)$'s are free variables. So we are reduced to proving the Lemma for $\mathcal{X}^{\text{univ}}$. If X and Y are short for the column vectors (X_1, \dots, X_d) and (Y_1, \dots, Y_d) respectively, the formal group law on $\mathcal{X}^{\text{univ}}$ is determined by

$$F_V(X, Y) = f_V^{-1}(f_V(X) + f_V(Y)), \quad \text{with } f_V(X) = \sum_{i=0}^{\infty} a_i(V) X^{p^i},$$

where $a_i(V)$'s are certain $d \times d$ matrices with coefficients in $\mathbf{Q}_p[V]$ with $a_1(V)$ invertible, X^{p^i} is short for $(X_1^{p^i}, \dots, X_d^{p^i})$, and f_V^{-1} is the unique d -tuple of power series in (X_1, \dots, X_d) with coefficients in $\mathbf{Q}_p[V]$ such that $f_V^{-1} \circ f_V = 1$ [Haz78, 10.4]. We note that $F_V(X, Y)$ is a d -tuple of power series with coefficient in $\mathbf{Z}_p[V]$, although $f_V(X)$ has coefficients in $\mathbf{Q}_p[V]$ [Haz78, 10.2(i)]. Via approximation by integers, we see easily that the multiplication by an element $\xi \in \mathbf{Z}_p$ can be well defined as $[\xi](X) = f_V^{-1}(\xi f_V(X))$. This proves (i). Statement (ii) is an immediate consequence of the fact that $f_V(X)$ involves just p -powers of X . \square

Proposition 1.5. *Let $G = \text{Spec}(A)$ be a connected finite and flat group scheme over \mathcal{O}_K of order a power of p . Then there exists a presentation of A of type (1.2.1) such that the defining equations f_i for $1 \leq i \leq d$ have the form*

$$f_i(X_1, \dots, X_d) = \sum_{|\underline{n}| \geq 1}^{\infty} a_{i, \underline{n}} X^{\underline{n}} \quad \text{with } a_{i, \underline{n}} = 0 \text{ if } (p-1) \nmid (|\underline{n}| - 1),$$

where $\underline{n} = (n_1, \dots, n_d) \in (\mathbf{Z}_{\geq 0})^d$ are multi-indexes, $|\underline{n}| = \sum_{j=1}^d n_j$, and $X^{\underline{n}}$ is short for $\prod_{j=1}^d X_j^{n_j}$.

Proof. By a theorem of Raynaud [BBM82, 3.1.1], there is a projective abelian variety V over \mathcal{O}_K , and an embedding of group schemes $j : G \hookrightarrow V$. Let \mathcal{X} be the formal completion of V along its unit section. This is a formal group over \mathcal{O}_K . Since G is connected, then j induces an embedding $i : G \hookrightarrow \mathcal{X}$. We denote by \mathcal{Y} the quotient of \mathcal{X} by G , and by $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ the canonical isogeny. Let (X_1, \dots, X_d) (resp. (Y_1, \dots, Y_d)) be parameters of \mathcal{X} (resp. \mathcal{Y}) satisfying the lemma above. The isogeny ϕ is thus given by

$$(X_1, \dots, X_d) \mapsto (f_1(X_1, \dots, X_d), \dots, f_d(X_1, \dots, X_d)),$$

where $f_i = \sum_{|\underline{n}| \geq 1} a_{i, \underline{n}} X^{\underline{n}} \in \mathcal{O}_K[[X_1, \dots, X_d]]$. Since for any $(p-1)$ -th root of unity $\zeta \in \mathbf{Z}_p$ we have $f_i(\zeta X_1, \dots, \zeta X_d) = \zeta f_i(X_1, \dots, X_d)$, it's easy to see that $a_{i, \underline{n}} = 0$ if $(p-1) \nmid (|\underline{n}| - 1)$. □

1.6. Proof of Theorem 0.1. Let $H = G^{0+}$ be the connected component of G . By 1.3(i), we have $G^a = H^a$ for $a \in \mathbf{Q}_{>0}$. On the other hand, from the exact sequence of group schemes $0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$, it follows that the sequence of finite \mathcal{O}_K -modules

$$0 \rightarrow \omega_{G/H} \rightarrow \omega_G \rightarrow \omega_H \rightarrow 0$$

is exact. Since G/H is étale, we have $\omega_{G/H} = 0$ and hence $\deg(G) = \deg(H)$. Up to replacing G by H , we may assume that $G = \text{Spec}(A)$ is connected.

We choose a presentation of A as in Prop. 1.5 so that we have an isomorphism of \mathcal{O}_K -algebras

$$A \simeq \mathcal{O}_K[[X_1, \dots, X_d]] / (f_1, \dots, f_d)$$

where

$$f_i(X_1, \dots, X_d) = \sum_{j=1}^d a_{i,j} X_j + \sum_{|\underline{n}| \geq p} a_{i, \underline{n}} X^{\underline{n}}.$$

Then we have

$$\Omega_{A/\mathcal{O}_K}^1 \simeq \left(\bigoplus_{i=1}^d A dX_i \right) / (df_1, \dots, df_d).$$

Since $\omega_G \simeq e^*(\Omega_{A/\mathcal{O}_K}^1)$, where e is the unit section of G , we get

$$\omega_G \simeq \left(\bigoplus_{i=1}^d \mathcal{O}_K dX_i \right) / \left(\sum_{1 \leq j \leq d} a_{i,j} dX_j \right)_{1 \leq i \leq d}.$$

In particular, if U denotes the matrix $(a_{i,j})_{1 \leq i,j \leq d}$, then we have $\deg(G) = v_\pi(\det(U))$.

For any rational number λ , we denote by $\mathbf{D}^d(0, |\pi|^\lambda)$ (*resp.* $\mathbb{D}^d(0, |\pi|^\lambda)$) the rigid analytic closed (*resp.* open) disk of dimension d over K consisting of points (x_1, \dots, x_d) with $v_\pi(x_i) \geq \lambda$ (*resp.* $v_\pi(x_i) > \lambda$) for $1 \leq i \leq d$; we put $\mathbf{D}^d(0, 1) = \mathbf{D}^d(0, |\pi|^0)$ and $\mathbb{D}^d(0, 1) = \mathbb{D}^d(0, |\pi|^0)$. Let $a > \frac{p}{p-1} \deg(G)$ be a rational number, X^a be the a -th tubular neighborhood of G with respect to the chosen presentation. By (1.2.2), we have a cartesian diagram of rigid analytic spaces

$$(1.6.1) \quad \begin{array}{ccc} X^a & \hookrightarrow & \mathbb{D}^d(0, 1) \\ \downarrow \mathbf{f} & & \downarrow \mathbf{f} = (f_1, \dots, f_d) \\ \mathbf{D}^d(0, |\pi|^a) & \hookrightarrow & \mathbb{D}^d(0, 1), \end{array}$$

where horizontal arrows are inclusions, and $\mathbf{f}(y_1, \dots, y_d) = (f_1(y_1, \dots, y_d), \dots, f_d(y_1, \dots, y_d))$. Let X_0^a be the connected component of X^a containing 0. By the discussion below (1.2.2), we just need to prove that 0 is the only zero of the f_i 's contained in X_0^a .

Let $V = (b_{i,j})_{1 \leq i,j \leq d}$ be the unique $d \times d$ matrix with coefficients in \mathcal{O}_K such that $UV = VU = \det(U)I_d$, where I_d is the $d \times d$ identity matrix. If \mathbf{A}_K^d denotes the d -dimensional rigid affine space over K , then V defines an isomorphism of rigid spaces

$$\mathbf{g} : \mathbf{A}_K^d \rightarrow \mathbf{A}_K^d; \quad (x_1, \dots, x_d) \mapsto \left(\sum_{j=1}^d b_{1,j}x_j, \dots, \sum_{j=1}^d b_{d,j}x_j \right).$$

It's clear that $\mathbf{g}(\mathbb{D}^d(0, 1)) \subset \mathbb{D}^d(0, 1)$, so that \mathbf{f} is defined on $\mathbf{g}(\mathbb{D}^d(0, 1))$. The composite morphism $\mathbf{f} \circ \mathbf{g} : \mathbb{D}^d(0, 1) \rightarrow \mathbb{D}^d(0, 1)$ is given by

$$(1.6.2) \quad (x_1, \dots, x_d) \mapsto (\det(U)x_1 + R_1, \dots, \det(U)x_d + R_d),$$

where $R_i = \sum_{|\underline{n}| \geq p} a_{i,\underline{n}} \prod_{j=1}^d (\sum_{k=1}^d b_{j,k}x_k)^{n_j}$ involves only terms of order $\geq p$ for $1 \leq i \leq d$. For $1 \leq i \leq d$, we have basic estimations

$$(1.6.3) \quad v_\pi(\det(U)x_i) = \deg(G) + v_\pi(x_i) \quad \text{and} \quad v_\pi(R_i) \geq p \min_{1 \leq j \leq d} \{v_\pi(x_j)\}.$$

Lemma 1.7. *For any rational number $a > \frac{p}{p-1} \deg(G)$, the map \mathbf{g} induces an isomorphism of affinoid rigid spaces*

$$\mathbf{g} : \mathbf{D}^d(0, |\pi|^{a-\deg(G)}) \xrightarrow{\sim} X_0^a.$$

Assuming this Lemma for a moment, we can complete the proof of 0.1 as follows. Consider the composite

$$\mathbf{h} = \mathbf{f} \circ \mathbf{g}|_{\mathbf{D}^d(0, |\pi|^{a-\deg(G)})} : \mathbf{D}^d(0, |\pi|^{a-\deg(G)}) \xrightarrow{\sim} X_0^a \hookrightarrow X^a \xrightarrow{\mathbf{f}} \mathbf{D}^d(0, |\pi|^a).$$

In order to complete the proof of 0.1, we just need to prove that the inverse image $\mathbf{h}^{-1}(0) = \{0\}$. Let (x_1, \dots, x_d) be a point of $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$, and $(z_1, \dots, z_d) = \mathbf{h}(x_1, \dots, x_d)$. We may assume $v_\pi(x_1) = \min_{1 \leq i \leq d} \{v_\pi(x_i)\}$. We have $v_\pi(x_1) \geq a - \deg(G) > \frac{1}{p-1} \deg(G)$ by the assumption on a . It follows thus from (1.6.3) that

$$v_\pi(R_1) \geq pv_\pi(x_1) > \deg(G) + v_\pi(x_1) = v_\pi(\det(U)x_1).$$

Hence, we deduce from (1.6.2) that $v_\pi(z_1) = \deg(G) + v_\pi(x_1)$. In particular, $z_1 = 0$ if and only if $x_1 = 0$. Therefore, we have $\mathbf{h}^{-1}(0) = \{0\}$. This achieves the proof of Theorem 0.1.

Proof of 1.7. Let ϵ be any rational number with $0 < \epsilon < \frac{p-1}{p}a - \deg(G)$. We will prove that

$$\mathbf{D}^d(0, |\pi|^{a-\deg(G)}) = \mathbf{D}^d(0, |\pi|^{a-\deg(G)-\epsilon}) \cap \mathbf{g}^{-1}(X^a).$$

This will imply that $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$ is a connected component of $\mathbf{g}^{-1}(X^a)$. Since $\mathbf{g} : \mathbf{A}_K^d \rightarrow \mathbf{A}_K^d$ is an isomorphism, the lemma will follow immediately.

We prove first the inclusion “ \subset ”. It suffices to show $\mathbf{g}(\mathbf{D}^d(0, |\pi|^{a-\deg(G)})) \subset X^a$. Let (x_1, \dots, x_d) be a point of $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$. By (1.6.1), we have to check that $(z_1, \dots, z_d) = \mathbf{f}(\mathbf{g}(x_1, \dots, x_d))$ lies in $\mathbf{D}^d(0, |\pi|^a)$. We get from (1.6.3) that $v_\pi(\det(U)x_i) = \deg(G) + v_\pi(x_i) \geq a$ and $v_\pi(R_i) \geq p(a - \deg(G))$. As $a > \frac{p}{p-1} \deg(G)$, we have $v_\pi(R_i) > a$. It follows from (1.6.2) that

$$v_\pi(z_i) \geq \min\{v_\pi(\det(U)x_i), v_\pi(R_i)\} \geq a.$$

This proves (z_1, \dots, z_d) is contained in $\mathbf{D}^d(0, |\pi|^a)$, hence we have $\mathbf{g}(\mathbf{D}^d(0, |\pi|^{a-\deg(G)})) \subset X^a$.

To prove the inclusion “ \supset ”, we just need to verify that every point in $\mathbf{D}^d(0, |\pi|^{a-\deg(G)-\epsilon})$ but outside $\mathbf{D}^d(0, |\pi|^{a-\deg(G)})$ does not lie in $\mathbf{g}^{-1}(X^a)$. Let (x_1, \dots, x_d) be such a point. We may assume that

$$(1.7.1) \quad a - \deg(G) - \epsilon \leq v_\pi(x_1) < a - \deg(G) \quad \text{and} \quad v_\pi(x_i) \geq a - \deg(G) - \epsilon \quad \text{for } 2 \leq i \leq d.$$

Let $(z_1, \dots, z_d) = (\det(U)x_1 + R_d, \dots, \det(U)x_d + R_d)$ be the image of (x_1, \dots, x_d) under the composite $\mathbf{f} \circ \mathbf{g}$. According to (1.6.1), the proof will be completed if we can prove that (z_1, \dots, z_d) is not in $\mathbf{D}^d(0, |\pi|^a)$. From (1.6.3) and (1.7.1), we get $v_\pi(\det(U)x_1) = \deg(G) + v_\pi(x_1) < a$ and $v_\pi(R_1) \geq p(a - \deg(G) - \epsilon)$. Thanks to the assumption on ϵ , we have $p(a - \deg(G) - \epsilon) > a$, so $v_\pi(z_1) = v_\pi(\det(U)x_1) < a$. This shows that (z_1, \dots, z_d) is not in $\mathbf{g}^{-1}(X^a)$, hence the proof of the lemma is complete. \square

2. APPLICATIONS TO CANONICAL SUBGROUPS

In this section, we suppose the fraction field K has characteristic 0 and the residue field k is perfect of characteristic $p \geq 3$. Let e be the absolute ramification index of \mathcal{O}_K . For any rational number $\epsilon > 0$, we denote by $\mathcal{O}_{K,\epsilon}$ the quotient of \mathcal{O}_K by the ideal consisting of elements with p -adic valuation greater or equal than ϵ .

2.1. First we recall some results on the canonical subgroups according to [AM04], [Ti06] and [Fa09]. Let $v_p : \mathcal{O}_K/p \rightarrow [0, 1]$ be the truncated p -adic valuation (with $v_p(0) = 1$). Let G be a truncated Barsotti-Tate group of level $n \geq 1$ non-étale over \mathcal{O}_K , $G_1 = G \otimes_{\mathcal{O}_K} (\mathcal{O}_K/p)$. The Lie algebra of G_1 , denoted by $\text{Lie}(G_1)$ is a finite free \mathcal{O}_K/p -module. The Verschiebung homomorphism $V_{G_1} : G_1^{(p)} \rightarrow G_1$ induces a semi-linear endomorphism φ_{G_1} of $\text{Lie}(G_1)$. We choose a basis of $\text{Lie}(G_1)$ over \mathcal{O}_K/p , and let U be the matrix of φ under this basis. We define the Hodge height of G , denoted by $h(G)$, to be the truncated p -adic valuation of $\det(U)$. We note that the definition of $h(G)$ does not depend on the choice of

U. The Hodge height of G is an analog of the Hasse invariant in mixed characteristic, and we have $h(G) = 0$ if and only if G is ordinary.

Theorem 2.2 ([Fa09] Théo. 4). *Let G be a truncated Barsotti-Tate group of level 1 over \mathcal{O}_K of dimension $d \geq 1$ and height h . Assume $h(G) < \frac{1}{2}$ if $p \geq 5$ and $h(G) < 1/3$ if $p = 3$.*

(i) *For any rational number $\frac{ep}{p-1}h(G) < a \leq \frac{ep}{p-1}(1 - h(G))$, the finite flat subgroup G^a of G given by the Abbes-Saito filtration has rank p^d .*

(ii) *Let C be the subgroup $G^{\frac{ep}{p-1}(1-h(G))}$ of G . We have $\deg(G/C) = eh(G)$.*

(iii) *The subgroup $C \otimes \mathcal{O}_{K,1-h(G)}$ coincides with the kernel of the Frobenius homomorphism of $G \otimes \mathcal{O}_{K,1-h(G)}$. Moreover, for any rational number ϵ with $\frac{h(G)}{p-1} < \epsilon \leq 1 - h(G)$, if H is a finite and flat closed subgroup of G such that $H \otimes \mathcal{O}_{K,\epsilon}$ coincides with the kernel of Frobenius of $G \otimes \mathcal{O}_{K,\epsilon}$, then we have $H = C$.*

The subgroup C in this theorem, when it exists, is called the *canonical subgroup* (of level 1) of G .

Remark 2.3. (i) The conventions here are slightly different from those in [Fa09]. The Hodge height is called Hasse invariant in *loc. cit.*, while we choose to follow the terminologies in [AM04] and [Ti06]. Our index of Abbes-Saito filtration and the degree of G are e times those in [Fa09].

(ii) Statement (iii) of the theorem is not explicitly stated in [Fa09, Théo. 4], but it's an easy consequence of *loc. cit.* Prop. 11.

For the canonical subgroups of higher level, we have

Theorem 2.4 ([Fa09] Théo. 6). *Let G be a truncated Barsotti-Tate group of level n over \mathcal{O}_K of dimension $d \geq 1$ and height h . Assume $h(G) < \frac{1}{3^n}$ if $p = 3$ and $h(G) < \frac{1}{2p^{n-1}}$ if $p \geq 5$.*

(i) *There exists a unique closed subgroup of G that is finite and flat over \mathcal{O}_K and satisfies*

- $C_n(\overline{K})$ is free of rank d over $\mathbf{Z}/p^n\mathbf{Z}$.
- For each integer i with $1 \leq i \leq n$, let C_i be the scheme theoretic closure of $C_n(\overline{K})[p^i]$ in G . Then the subgroup $C_i \otimes \mathcal{O}_{K,1-p^{i-1}h(G)}$ coincides with the kernel of the i -th iterated Frobenius of $G \otimes \mathcal{O}_{K,1-p^{i-1}h(G)}$.

(ii) *We have $\deg(G/C_n) = \frac{e(p^n-1)}{p-1}h(G)$.*

The subgroup C_n in the theorem above is called the canonical subgroup of level n of G . Fargues actually proves that C_n is a certain piece of the Harder-Narasimhan filtration of G . The aim of this section is to show that C_n appears also in the Abbes-Saito filtration.

Theorem 2.5. *Let G be a truncated Barsotti-Tate group of level n over \mathcal{O}_K satisfying the assumptions in 2.4, and C_n be its canonical subgroup of level n . Then for any rational number a satisfying $\frac{ep(p^n-1)}{(p-1)^2}h(G) < a \leq \frac{ep}{p-1}(1 - h(G))$, we have $G^a = C_n$.*

Proof. We proceed by induction on n . If $n = 1$, the theorem is 2.2(i). We suppose $n \geq 2$ and the theorem is valid for truncated Barsotti-Tate groups of level $n - 1$. For each integer i with $1 \leq i \leq n$, let G_i denote the scheme theoretic closure of $G(\overline{K})[p^i]$

in G , and C_i the scheme theoretic closure of $C_n(\overline{K})[p^i]$ in C_n . By Theorem 2.4(i), it's clear that C_i is the canonical subgroup of level i of G_i . Let a be a rational number with $\frac{ep(p^n-1)}{(p-1)^2}h(G) < a \leq \frac{ep}{p-1}(1-h(G))$. By the induction hypothesis and the functoriality of Abbes-Saito filtration 1.3(ii), we have $C_{n-1}(\overline{K}) = G_{n-1}^a(\overline{K}) \subset G^a(\overline{K})$, and the image of $G^a(\overline{K})$ in $G_1(\overline{K})$ is exactly $C_1(\overline{K}) = G_1^a(\overline{K})$. Note that we have a commutative diagram of exact sequences of groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{n-1}(\overline{K}) & \longrightarrow & C_n(\overline{K}) & \longrightarrow & C_1(\overline{K}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & G_{n-1}(\overline{K}) & \longrightarrow & G(\overline{K}) & \xrightarrow{\times p^{n-1}} & G_1(\overline{K}) \longrightarrow 0, \end{array}$$

where vertical arrows are natural inclusions. So we have $C_n(\overline{K}) \subset G^a(\overline{K})$. On the other hand, Theorems 0.1 and 2.4(ii) imply that $(G/C_n)^a(\overline{K}) = 0$ as $a > \frac{ep(p^n-1)}{(p-1)^2}h(G) = \frac{p}{p-1} \deg(G/C_n)$. Therefore, we get $G^a(\overline{K}) \subset C_n(\overline{K})$ by 1.3(ii). This completes the proof. \square

REFERENCES

- [AM04] A. ABBES and A. MOKRANE, Sous-groupes canoniques et cycles évanescents p -adiques pour les variétés abéliennes, *Publ. Math. Inst. Hautes Étud. Sci.* **99** (2004), 117-162.
- [AS02] A. ABBES and T. SAITO, Ramification of local fields with imperfect residue fields, *Am. J. Math.* **124** (2002), 879-920.
- [AS03] A. ABBES and T. SAITO, Ramification of local fields with imperfect residue fields II, *Doc. Math. Extra Volume: Kazuya Kato's Fiftieth Birthday* (2003), 5-72.
- [BBM82] P. BERTHELOT, L. BREEN, and W. MESSING, *Théorie de Dieudonné Cristalline II*, Lecture Notes in Math. **930**, Springer-Verlag, (1982).
- [Fa07] L. FARGUES, La filtration de Harder-Narasimhan des schémas en groupes finis et plats, preprint in 2007, to appear in *J. für die Reine und Angewandte Math.*
- [Fa09] L. FARGUES (avec la collaboration de Yichao TIAN), La filtration canonique des points de torsion des groupes p -divisibles, preprint in 2009, available at the author's homepage.
- [Hat06] S. HATTORI, Ramification of a finite flat group scheme over a local field, *J. Number Theory*, **118**, Issue 2, 145-154.
- [Haz78] M. HAZEWINKEL, *Formal groups and applications*, Academic Press, (1978).
- [Ti06] Y. TIAN, Canonical subgroup of Barsotti-Tate groups, arXiv:math/0606059, to appear in *Ann. Math.*

MATHEMATICS DEPARTMENT, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ, 08544, USA
E-mail address: yichaot@princeton.edu